On Fisher's Reproductive Value and Lotka's Stable Population

Carlos Galindo
Consejo Nacional de Población, México
cgalindo@colmex.mx
carlosglx@gmail.com

Abstract

The main purpose of this paper is to finally debunk the idea that Fisher implied the reproductive value (RV) of the first age group *must* be equated to 1 (other valuations answer different research questions). Fisher (1927) and other researchers stated that dissimilar generation times must be considered when comparing subpopulations (confront with Keyfitz and Caswell 2005). Fisher's (1927) original paper also points out, in a most unusual way, the links with Lotka's stable theory. RV has *never* been presented along with proper interpretation of stable concepts used in its development (see Fisher 1927, 1930); revisiting Lotka's thermodynamic assumptions will prove helpful to avoid confusions and misinterpretations of RV and related concepts as stable equivalent value and population momentum.

Keywords

Reproductive value, Stable population, Population momentum, Thermodynamics, reproduction, Fisher, Lotka.

On Fisher's Reproductive Value and Lotka's Stable Population

> "The History of Science has suffered greatly from the use by teachers of second-hand material, and the consequent obliteration of the circumstances and the intellectual atmosphere in which the great discoveries of the past were made. A first-hand study is always instructive, and often ... full of surprises."
>
> - *Ronald Fisher*

INTRODUCTION

When reading an earlier version of this paper, an anonymous referee wrote that since Leslie's (1945, 1948) work "it is clear that reproductive value in the discrete-time formulation is an eigenvector of the projection matrix… Since the components of these vectors/functions can only be measured relative to each other, it is purely a matter of convenience how to define v_0. I cannot see that this question needs lengthy discussion". And I would agree with this comment if I had not listen and read many others arguing: "That is true, but Fisher also stated that at birth, everyone gets one life ($v_0 = 1$) and must pay back the debt with an interest" (the concept v_0 will be explained in the following sections). As a matter of fact, Fisher and other researchers were fully aware that defining $v_0 = 1$ is not suitable for population comparisons because *generation time* must be taken into account; however, several influential books still use this inappropriate conventional definition when trying to compare reproductive values of different subpopulations (e.g., see Keyfitz and Flieger 1971; Keyfitz and Caswell 2005).

The main purpose of this paper is to finally debunk the idea that Fisher implied that v_0 *must* be equated to 1; and also to emphasize that Fisher (and other researchers) realized that this conventional valuation is not appropriate for population comparisons. This is of importance because, as it is shown at the end of this paper, the arbitrary definition $v_0 = 1$ can lead to contradictory results when compared to a more suitable convention that considers generation time and equates total reproductive value to the 'number of heads alive in the *steady state* of the population'.

Fisher's (1927) original paper links RV to Lotka's stable population theory and other 'stable' concepts as Keyfitz's population momentum. In an earlier version of this paper I just noted that RV and related concepts should be interpreted through the guidelines given by thermodynamic assumptions but one reviewer wrote, "The concepts of reproductive value, stable equivalent and momentum are all different… This is not clear. Could you briefly state why it is based on the theory of thermodynamics? Only because of the concept of steady state?" Being this reviewer an eminent professor who has worked for several years with RV and stable theory, I realized that it was not a well-known fact among demographers that stable population theory is entirely based on thermodynamics. So I added a brief section about proper interpretation of thermodynamic assumptions. However, this section has aroused controversy among reviewers, as one wrote "Let me mention that an important feature of this article is showing the connection of demography to two quite disparate subjects: thermodynamics and monetary investment", but another one labeled it as "a rambling discussion of many aspects of stable population theory". In mathematics, and other disciplines as physics, a tool cannot be presented without the assumptions used in its development. I cannot see why social sciences, particularly demography, should be any different. As Reproductive Value was *never* introduced along with the thermodynamic assumptions used in its development (see Fisher 1928, 1930), I believe that it is necessary to briefly explain the proper interpretation of these assumptions.

REPRODUCTIVE VALUE

Caswell (2001:92) cites Fisher's (1930) explanation "that the present value of the future offspring of a person of age x is 'easily seen to be given by the equation' "

$$\frac{v_x}{v_0} = \frac{e^{rx}}{l_x} \int_x^{\infty} e^{-ru} l_u b_u du$$

Then Caswell adds "With all due respect to Fisher, I have yet to meet anyone who finds this equation 'easily seen'". Caswell is right; it is not easy to achieve intuitive understanding of Fisher's Reproductive Value (RV). However, James Crow (2000, 2002) recently made a very interesting finding which will help to make Fisher's equation more 'easily seen':

> Fisher (1999) [reprint of 1930] defined reproductive value at age x, v_x, only as a number relative to an arbitrary value of one at age zero, v_0. I, and undoubtedly others, had often wondered why Fisher did not give v_x an absolute meaning, and in fact Keyfitz (1968) later did just that (see also Samuelson 1977). Keyfitz (1968, p. 57) called the reproductive value-weighted total population the 'stable equivalent population'. I had always attributed this to Keyfitz, but in glancing through Fisher's collected papers some years ago, I noticed a throw-away article published in *Eugenics Review* (Fisher 1927). Here, of all places, Fisher gives a definition for the reproductive value at age zero:
>
> $$v_0 = 1/\overline{b}\overline{x}_r$$
>
> in which b is the birth rate and x_r is the age of reproduction, both averages being for a population at age stability.
>
> When this value is used, the total reproductive value merges smoothly into the population number as the age structure stabilizes (Crow

> 1979). It is the same as Keyfitz's stable equivalent population. Why Fisher failed to mention this in his 1930 book, written sometime after 1927, is a complete puzzle; it is almost as if Fisher was determined to confuse. In particular, generations of readers of his 1930 book have found the graph on page 28 confusing. This plots the reproductive values of Australian women at about 1911 as ordinate against age. The value at age zero is clearly not unity, as Fisher had led the reader to expect. This discrepancy is finally explained in the variorum edition (Fisher 1999 p. 302) where Fisher, in a letter to C. G. Darwin, dated 16 July 1930, writes: 'I am sorry about v_0. It is unity by definition on page 27 but when I came to make the graph I introduced a factor so as to make the total number of heads in the population in its steady state equal to the total value of such a population. That made v_0 a trifle over 2,' as is apparent from the graph. No wonder the graph is confusing. Incidentally when he invented the idea of the stable equivalent population, Keyfitz was not aware of Fisher's discovery, long before in 1927. Keyfitz was not alone. (Crow 2002:1315)

The goal of Crow's paper is to clarify Fisher's conjecture about natural selection from the "usual elegant obscurity" (Crow 2002:1314) of Fisher's writings (this conjecture was named the 'Fundamental Theorem of Natural Selection' by Fisher himself). After writing the paragraphs quoted above, Crow moves away from reproductive value and follows the development of Fisher's conjecture. However there are still some other things to clarify about Crow's finding (for example, Keyfitz's did not "invented the idea of the stable equivalent population", he just coined that term when studying the relationship between Fisher's discrete RV and Lotka's continuous stable 'standard' population). Revising Fisher's (1927) original RV formulation will be instructive for three reasons: reproductive value is still an abstract concept and there is no common agreement about its demographic uses and interpretations; the original formulation of RV sheds some light on how Fisher intended this tool to be used; and this paper also gives some clues about the relation of RV with Lotka's stable theory. I will go through Fisher's original formulation in order to show these points.

FISHER'S 'ORIGINAL' FORMULATION OF RV

It is still commonly believed that Fisher (1930) originally proposed the concept of reproductive value in his book *The genetical theory of natural selection* as a tool for assessing the genetic contribution of one individual to a future population, and that afterwards this concept was adapted for comparisons among different populations. However, the concept of RV was originally developed with the purpose of comparing 'potential' reproduction among human populations. In 1927 Fisher wrote a paper for *Eugenics Review* called "The Actuarial Treatment of Official Birth Records", in this paper he proposed for the first time the idea of a reproductive value associated to a certain population. His aim was to develop a technique for establishing the optimal reproduction of the 'working class' given different occupational groups. The calculations were based on men and their sons because Fisher assumed that sons had to remain in the occupational group of their fathers and that women should not enter any occupational group (as the rest of the eugenist movement, Fisher held an exaggerated view of the importance of genes in determining social traits; in this paper Fisher develops the concept of reproductive value based on mistaken beliefs about the inheritance of mental and working capabilities), but he also pointed that calculations should be made on women and their daughters for regional comparisons of subpopulations.

Although RV was formulated according to mistaken assumptions about human hereditability, this tool has proven to be useful in Biology and Demography. If we are able to separate the RV statistical tool from the faulty assumptions of Fisher (1927), then his statistical definitions make sense; and, as Crow (2002) points out, Fisher's original paper provides better and more complete explanations of RV than Fisher's (1930) book. Additionally, this paper also provided, in a most unusual way, the links between RV and Lotka's stable population theory.

After reading Fisher 1927, Lotka (1927) sent a letter to *Eugenics Review* where he shows that he had already written the main ideas of Fisher's publication and that several paragraphs can be seen as copies of Lotka's previous work (this is not really a surprise given Fisher's practice of not attributing proper credit to preceding ideas, for another example see Stigler 2005; despite there is no single citation in Fisher's paper, many researchers were also working with the renewal equation, other example is Haldane 1927; the letter of Lotka is shown in the Annex). Lotka states "That the actuarial principle discussed by Dr. Fisher may be used as an instrument to measure effective fertility is fully set forth, with numerical application, in my paper: *The Measure of Net Fertility, Journal of the Washington Academy of Science*. December, 1925, page 469. A particular detailed application, with much numerical elaboration, of the principle under discussion, has been given by Dublin and Lotka in the paper referred to above, *The True Rate of Natural Increase*. Among other things brought out and illustrated by actual numerical computations in this paper is the separate application of the principle to the female and the male population, a point to which Dr. Fisher also refers". To this letter Fisher merely replied (also in Lotka 1927) that he was not aware of Lotka's work and that "Evidently the only absolutely novel suggestion in my article lies in the estimation of a definite 'reproduction value' for each age of life". The 'novel' suggestion that Fisher claims about a value for each age of life applied to vital records was made, for the first time, by the statistician and epidemiologist William Farr (who worked almost 40 years in the General Register Office of England). William Farr first stated his concept of 'value' for each age of life when looking at economical contributions and Vital Statistics (Farr wrote a book called *Vital Statistics* which is out of print, but the traces of his work can be followed in Humphreys 1885). Later, Lotka (1944) explained Farr's work, and it will be useful to briefly look at his explanation because it shows the 'economical' or 'financial' rationale behind reproductive value. On the other hand Lotka's work is entirely based in Thermodynamics theory (read Lotka 1998: chaps. 1. On Evolution in Organic and Inorganic Systems, 2. On the Direction of Time, 3. On Energetics and Uncertainty; also Lotka 1969:9-40) and briefly revising his work will also be useful to clarify the population comparisons intended by Fisher (1927). So, from its very origin, RV is a 'multidisciplinary' tool, and trying to understand the meaning of the concepts used in its development will be helpful for understanding the tool itself.

I will start by following Fisher's (1927) formulation of RV. First he explains that the 'expectation of offspring' of a newly born child can be estimated by "the average number of live births already conceived at each age" by already living persons (1927:104). Today we refer at this indicator as Net Reproduction Rate (considering only one sex for calculating the birth rate).

$$R_0 = \int_0^\infty l_x b_x dx \quad , \tag{1}$$

where x denotes age, l and b stand for age specific survival and fertility respectively. Here it should be noted that using data from a life table (l and b) implicitly means to accept the *stationary population* assumption; the importance of this clarification will become evident in the section about stable population theory.

Fisher then explains that the *Malthusian parameter* (stable growth rate) derives from Malthus' analogy of population increase with compound interest: in this metaphor the date of repayment of a debt is important for the calculation of the rate of interest, and in order to find the appropriate rate in the analogous problem of human increase we must reduce the future children, whose advent we are expecting, to their *present value* (of new born child) equal to unity. This provides an exact measure of the exponential rate of increase of the population (geometrical in discrete case). Fisher also points out that Malthus scarcely considered that a decreasing population must be represented by a negative rate of interest, which will occur when the statistical mean of same sex offspring is less than unity. Fisher never mentions it, but the resulting expression is known as the Euler-Lotka equation, in honour of the mathematician that developed it and the ecologist who gave it a biological and demographical use.

$$\int_0^\infty e^{-rx} l_x b_x dx = 1 \quad , \tag{2}$$

where r represents the stable growth rate, also referred as intrinsic rate or Malthusian parameter.

The next step is to add this 'value' to an age group. Fisher claims that the *present value* of the future progeny of an age group, as defined in financial mathematics with the rate of compound interest indicated above, will provide an exact measure of the present value of individuals in the group considered for procreating future generations. He also says that the present value of each individual will increase with age as he (or she) escapes the dangers of infant and child mortality, it will reach its maximum at about 20, and will thereafter decrease as the time for procreation passes, whether such procreation has been realized or not. Then, when reproduction has ceased the reproductive value as a potential ancestor is obviously zero (in his later work on genetics Fisher 1930 referred to this concept as age-specific reproductive value).

$$v_x = \frac{1}{e^{-rx} l_x} \int_x^\infty e^{-ru} l_u b_u du \quad . \tag{3}$$

The first important clarification to RV is made by Fisher in his original paper: "The convention that unit value is to be ascribed to the newly born is open to no objection so long as we merely wish to compare the values of different age groups; on the other hand it is not suitable for the comparison of different populations" (Fisher 1927:106). The term *convention* or *conventional valuation* is a financial one, and it can be translated into plain English as 'you can use any number you want'. This means that some numbers might be more useful than others but there is no 'correct' number for ascribing the value of newborns. This very important point is not clearly stated in Crow (2002): there is no 'absolute' definition for the value of the newborns v_0, and this lack of absolute definition gives flexibility to the use of RV. In a private communication Crow explained "For me, although there may not be any 'correct' value, the value in which v_0 is the reciprocal of the product of the birth rate and the mean age of reproduction, both measured at the stable age distribution, is more natural than the others (I am still amazed that Fisher did not include this in his famous book, when he clearly knew it)." Apart from what can be seen as a more natural valuation, the fact is that what Fisher meant with *conventional valuation* is that there is no correct or absolute definition for v_0, and this is important to note it because it is still a confusing point. RV can be used differently according to different assumptions, or conventional valuations, about the number ascribed to the first age group v_0; but we should also be mindful that the results given by RV are highly dependent on the assumptions made about v_0 values and arbitrary population divisions.

Fisher (1927) refers briefly to the ideas of *conventional valuations*, *compounded interest* and *present value*; which indicate the original ideas behind the formulation of reproductive value. But in the original work of Farr this economical thinking is clearly stated. Lotka (1944) explains the original work of Farr (he cites Farr, W. *Journal Stat. Soc. London* 1853, but he gives no title of Farr's publication). Farr was initially trying to estimate the 'capitalized value of human earning capacity'. He equates the *capital value* of a wage-earner to the 'present worth' of his net future earnings. Let the *capital value* be denoted by v_x for a wage-earner of age x

$$v_x = \frac{l_0}{(1+i)^{-x} l_x} \sum_{u=x}^{\omega} (1+i)^{-u+1/2} L_u W_u \quad , \tag{4}$$

where l_x and L_x have the usual demographic meaning (survival estimations), W_x denotes the average of the earners' net annual earnings at age x to x+1, and i denotes the interest rate applied annually to the earnings.

Lotka explains that "Since l_0 is a purely arbitrary constant (the *radix* of the life table), we can arbitrarily put l_0=1. Then l_x and L_x, instead of numbers of individuals, represent corresponding proportions. It will simplify our formulae to adopt this convention" (Lotka 1944:10). And here we see again that the assigned value for the first age group (even if here is the radix) is only a *convention*, an *arbitrary assumption*, or in plain English, is just a matter of taste. Even if Farr's calculation is based on discrete time, it is easy to see that if we use birth rates instead of net earnings we obtain an 'annualized' reproductive value. Lotka notes that when the unit of study is the population (not the individuals) it is better to use an instantaneous rate of interest because "the population brings a continuous income, unlike a loan of money, which brings an income at finite intervals" (Lotka 1944:12). Let r be the interest rate compounded continuously, then the terms that adjust the age-population values $(1+i)^{-x}$ are replaced by e^{-rx}, the approximation made by L_x is no longer necessary, and also the approximation made with the net annual earnings at age x to x+1, denoted by W_x, is no longer necessary (the average annual earnings at age x is denoted by Lotka as w_x). Thus Farr's discrete equation 4 can be expressed in the continuous setting as

$$v_x = \frac{1}{e^{-rx} l_x} \int_x^{\infty} e^{-ru} l_u w_u du \ . \tag{5}$$

So the financial thinking behind reproductive value is as follows: the net reproductive rate, equation 1, can be regarded as a sum of money related to each age (earnings by age in Lotka's explanation or payments by age in Fisher's explanation); in order to compare the different amounts obtained at each age we need to 'move' them towards the same reference time point (age zero), this is done by adjusting the amounts with the prevailing interest rate (the intrinsic growth rate of the population); but the amount related to age zero is going to be our reference amount, so we assign an arbitrary value (a conventional valuation) to this first amount, which yields equation 2; finally, in order to make a fair comparison we need to standardize for the size of the population at each age, which is done dividing by l_x, but this size also has to be moved to the same reference time point $e^{-rx}l_x$. And so we obtain equations 3 and 5 (equation 5 is called the *present value* at age x of a continuous earning or payment stream of w_u beyond age x).

There are two important things to note from Lotka's explanations. The first important thing to note is that when using RV the unit of study is the population, so we cannot ascribe a reproductive value to single individuals. Since RV is an average it would be a mistake to state that a woman or a migrant has certain RV, we can only say that a population (or subpopulation made by an arbitrary division by age or other category) has defined values of RV. The second point is that RV as defined in equation 3 is a tool designed for comparisons of reproductive behaviour between different age groups: first it 'moves' age-specific values to the same reference point; then it standardizes these values by the size of the age groups and; finally it compares the standardized values against the arbitrary reference of the first age group. This last point may seem repetitive but is important to emphasize it because, as Crow (2002) mentions, from reading the book of Fisher (1930) many researchers believe that the reproductive value of age zero is 'defined' as the unity. This problem arose mainly from the metaphor used by Fisher of an acquired debt at birth of one life and its subsequent payment by age-contributions (which wrongly leads us to believe that he defined v_0=1 because of this 'one' life that we are supposedly granted), but the important term to note in the metaphor is the *convention* of the debt's value (and as we have seen from Lotka's explanation, the reproductive payments can also be seen as reproductive earnings, and instead of a debt we merely have the present value of such earnings). So once again, Fisher (1927) did not define v_0 as the unity, he explained that v_0 can be any number we want it to be (that is what *conventional valuation* or *arbitrary assumption* mean, that there is no absolute definition for v_0), and he proposed to use v_0=1 when we are interested in comparing RV of different ages within the same population (because it is easy to compare values against the unity but, strictly speaking, we can also use v_0=1000 or v_0=3.1416 and we would still be following Fisher's original formulation of RV). Later I will show that, as Crow (2002) also mentions, Fisher (1927) proposed other conventional valuations or arbitrary values for v_0 intended for other kind of comparisons. That is why the original presentation of the formula of RV made by Fisher (1927) includes the term v_0, and he just notes that v_0=1 if we want to assume this value for a newly born child.

$$v_x = \frac{v_0}{e^{-rx} l_x} \int_x^\infty e^{-ru} l_u b_u du \quad . \tag{6}$$

I will return to Fisher because he makes the next important clarification, when knowing the values to assign to each age we may evaluate the whole census population RV when the age distribution is also known. He adds that "The comparison of the *total values* of two census populations, unlike the comparison of the mere numbers, provides, when allowance has been made for migration, a simple measure of the population growth or decrease, which may be shown to coincide with the Malthusian rate of interest discussed above, or rather if it is changing, with its value averaged over the intercensal period" (Fisher 1927:106, italics on the original). He also notes that this ratio can be used to test whether the increase in the number of heads of the population is or is not sufficient to counterbalance the increasing average age (in an aging population). What Fisher calls the *total value* is given by

$$V = \int_0^\infty v_x n_x dx = \int_0^\infty \left(\frac{n_x v_0}{e^{-rx} l_x} \int_x^\infty e^{-ru} l_u b_u du \right) dx \quad , \tag{7}$$

where n denotes population numbers.

Some researchers refer to this concept as net reproductive value, total RV or population RV, but there is also some confusion about it because, once again, this explanation was not made in Fisher 1930. Stearns (1976) explains that Leslie defined the RV for the whole population as the sum of the age specific RV. In the continuous setting the total RV proposed by Leslie is

$$V^* = \int_0^\infty \frac{v_x}{v_0} dx = \int_0^\infty \left(\frac{1}{e^{-rx} l_x} \int_x^\infty e^{-ru} l_u b_u du \right) dx \quad . \tag{8}$$

Clearly the measure proposed by Leslie V^* is different from the one proposed by Fisher V. Leslie's V^* is merely the sum of the age-specific RV standardized by v_0, while Fisher's V is not standardized and it also takes into account population size and its age distribution. So Leslie's V^* can be seen as the Crude or Gross RV of the population, and Fisher's V can be seen as the Net RV of the population. And once again, the choice of using V^* or V depends on which kind of comparison we want to make.

Last but not least, Fisher makes a very important explanation about the use of RV (so important that it was the only one reported in Crow 2002): "The convention that unit value is to be ascribed to the newly born is open to no objection so long as we merely wish to compare the values of different age groups; on the other hand it is not suitable for the comparison of different populations. For this purpose a different convention will be more suitable, namely, that in a population in its steady state the total value ascribed to the population is equal to the total number of heads living" (Fisher 1927:106). So Fisher explains that when comparing different populations the convention of the unity for newborns is not suitable, instead is better to consider the net RV of the population V in its *steady state*, which would be equal to the number of individuals hypothetically alive. Therefore Fisher states that the arbitrary assumption of v_0=1 is not appropriate for comparing different populations, so he proposes another arbitrary assumption (which is the one emphasized by Crow)

$$v_0 = \frac{1}{\overline{b^s x_r^s}} \quad , \tag{9}$$

where the bar denotes mean, and the superscript *s* indicates stability or steady state.

Fisher explained the reason for this alternative valuation in terms of recruiting sons for their fathers' occupations: "If two occupations, for example, were each halving their numbers in each generation the rate of decrease *per annum*, and the corresponding need of recruitment, would be greater in that which had the shortest generation, and the sons born early should count for less than the sons born late" (Fisher 1927:105, italics in the original). Disregard Fisher's flawed ideas of working vocations, the reason for this new arbitrary valuation is that generation rate (stable birth rate) and generation time (stable mean age at childbearing) must be taken into account when comparing different populations. In other words, newborns from populations with different intrinsic birth rates and generation times should not be represented with the same RV. Interestingly enough, this alternative valuation overrides the critic to RV made by Hamilton (1966) about the lack of a measure of generation time which would serve as the unit of progress under natural selection; and perhaps more surprising, the way of taking into account generation time is the same solution proposed by Hamilton many years after Fisher's (1927) original paper. Because I have found this claim difficult to accept by some researchers, I will briefly explain it.

Is important to note that I am not talking about Hamilton's discussion about RV and senescence (Hamilton pointed out that Medawar 1957 misused RV in his theory of senesce; but that is a complete different topic of what is being discussed in this paper). Hamilton explained that "For an organism which reproduces repeatedly the concept of fitness is not so easily defined.

The expectation of offspring suffers from the objection that early births are worth more than late in an increasing population, and vice versa in a decreasing one, and that there is no single measure of generation time which will serve as the unit for progress under natural selection" (Hamilton 1966:96). Hamilton also noted that the expectations of births occurring after age *a* to persons chosen at age 0 and at age *a*, and the RV of each age group "are clearly similar except that in them [RV values] the births have been weighted in a particular way. As mentioned before, the weights are those necessary to correct for the different values of early and late births in a non-stationary population from the point of view of contribution to the population of the distant future" (Hamilton 1966:105). This means that RV fixes the problem about assigning proper weights to early and late births but the one about the measure of generation time still remains. Fortunately, Hamilton also proposed a solution "This is one of the parameters which can be considered as measuring the length of a generation, being the mean age of mothers at childbirth for all births occurring in the stable population" (Hamilton 1966:99). Interestingly enough, Leslie (1966) also proposed to use the mean age at childbirth in the stable population as a measure of the length of a generation.

Fisher also explained that this new valuation based upon the *steady state* allows the newborn child to count for more among long-lived people (aged population) than among the short-lived (young population). He also claimed that the comparison of the Net RV of the census population with the Net RV of the population in the *steady state* provides a simple index showing to what extent the actual populations are or are not of ages favourable to reproduction (to his statement it can be added that the problem of comparing different populations is closely related to comparing the same population over time). Fisher clearly noted that in order to make the *steady state* comparison the Euler-Lotka equation should be equated to an initial value named v_0, and v_0 must be chosen so that, in the *steady state*, the total reproductive value V is equal to the population size (equation 9). This is the reason why Fisher (1927) emphasized several times that the value of v_0 is just a convention and this is the reason why he presented the original formula of RV as equation 6. Given the importance of this new valuation, it is justified to ask the following questions: What is this *steady state* of the population? How is it related to actual populations? How should it be interpreted? More generally, what does it mean?

LOTKA'S STABLE POPULATION

There is one term that Fisher (1927) uses in the last part of his paper that will be fruitful to analyze because it inevitably leads to Lotka's stable population (Lotka 1927). The term *steady state* comes from the theory of Thermodynamics. By the time Fisher wrote his paper much confusion remained about the use of thermodynamic states to characterize dynamic processes (perhaps outside Physics this confusion still remains in our days). However by 1927 Lotka had already discussed thermodynamic concepts applied to human populations in several papers, and he had already set forward the terms of 'stable' age distribution, 'stable state' of the population or stable 'standard' population (the best example is his influential paper about the intrinsic growth rate, Lotka 1925). A small clarification to what Crow wrote, it was not Keyfitz who "invented the idea of the stable equivalent population" (Crow 2002:1315), the concept and the idea were already present in Lotka's constant *K*. Later Keyfitz merely proved some of the relations that exist between Lotka's continuous treatment of stable populations and Leslie and others discrete treatments (Goodman 1967); particularly he proved the relation between Fisher's RV and the size of Lotka's stable population, naming this relation 'stable equivalent population' or 'stable equivalent value'. In fact, Lotka worked on stable populations long before Fisher wrote about 'steady' states. Since 1911 Sharpe and Lotka were working in problems related to age distributions, and since 1921 Lotka started relating the age distribution problems with thermodynamic *moving equilibria*.

I will just sketch briefly the ideas behind *steady states*, for detailed and rigorous treatment please read Lotka (1939), or its recent English translation (Lotka 1998), also Zotin (1990) and Haynie (2001). According to the theory of thermodynamics over chemical reactions, systems can be classified in isolated, closed and open systems. Isolated systems exchange nothing with their surroundings (environment), while closed systems exchange only energy and open systems exchange energy and matter. Isolated systems tend towards equilibrium states, while closed and open systems tend towards non-equilibrium states often referred as steady states. “Non-equilibrium steady states of macroscopic systems, whether close to or far from equilibrium, share many of the features of equilibrium states… In fact, the only operational difference between an equilibrium state and a nonequilibrium steady state is that a flux of mass, momentum, or energy is being transported by the system” (Keizer 1984:1115). This is the reason why Lotka (1921) refers to the steady states as *moving equilibria*.

Starting from the assumption of closed systems Lotka (1939) discusses two important steady states for the population: if we assume that the population is a closed system (he names this assumption as *closed population*) then we expect the population to move towards two possible steady states (and the trivial 0 state), the stationary population (originally implied by the life table) and the stable population. These two moving equilibria result from the evolution of the closed system, but they are easier to understand through variations of the steady state assumption; according to Haynie (2001) in 1925 Haldane and Briggs put forth this assumption: in the steady state the rate of formation of the complex is equal to the rate of its decomposition, so the rate of formation is constant and its velocity is equal to zero. Therefore the stationary population is reached when the rate of formation of the complex (the birth rate) is equal to the rate of its decomposition (the death rate), so the rate of the reaction implied by these two rates (the population growth rate) is constant and equal to zero. The stable population is reached by relaxing the conditions of the stationary population, in this case the rates of formation and decomposition are held also constant but they are not equal, so the rate of reaction (the population growth rate) is constant but different from zero. In his original papers Lotka had no space to explain the thermodynamic basis of his stable population (for example his influential paper of 1925), but in his book (Lotka 1939) he fully explained his assumptions and his sources of inspiration. That non-equilibrium thermodynamics is the basis of the stable population theory is not a well known fact for two reasons: the book of Lotka (1939) was published in French and not until recently (1998) was it translated to English; furthermore, Lotka's book was originally published in two separate volumes, the first one dealt with thermodynamic assumptions and principles explanations, and the second one dealt with their application to human populations. So most of the demographers interested in the topic only acquired the second volume. Nowadays, fortunately, the Spanish (1969) and English (1998) translations include both volumes in the same publication.

There is one main problem with Lotka's assumptions; it will be fruitful to revise it for understanding stable population theory and the use of RV for comparing populations. Lotka is taking literally the closed system assumption for chemical reactions (exchange of energy but not of matter with the environment), so he says that the *closed population* assumption is a population closed to migration (he thought that migration was 'matter' exchange). However, even if population dynamics are somewhat dependent on chemical reactions, the population dynamics do not behave exactly as chemical processes; so the literal translation of chemical closed systems is not suitable for populations. The economic translation for the closed system assumption is far more appropriate (classic economic theory also has its basis in thermodynamics because some of its founding fathers were also physicists, for the complete story read Mirowski and Goodwin 1991, and for some of the mathematic relationships read Smith and Foley 2002); a closed system is translated into economic theory as 'with all the things being the same' or 'all other things being equal'. So the closed population is a *ceteris paribus* population (the environment remains constant, which includes constant natural, social, economic and cultural context). This formulation of the closed population assumption (*ceteris paribus* population) is the correct one, and this fact is easy to see when we realize that the steady assumption can include migration. In a closed population the rate of formation of the complex (birth and immigration rates) and the rate of decomposition (death and emigration rates) are equal, so the rate of formation (population growth rate) also remains constant and its velocity is equal to zero (and so the stationary age distribution is also reached, for the stable age distribution is only needed that birth, death and migration rates remain constant). Several researchers have discussed the fact that the 'no-migration' assumption of Lotka is not necessary for reaching stability; it is only necessary that migration rates remain constant (perhaps the best examples of this result are found in Feeney 1970 and Keyfitz 1971b, a longer list of researchers can be found in Cerone 1987). What is called in demography as the closed population assumption allows considering migration (which only affects the absorbing nature of the zero state), and more appropriately biologists refer to this assumption as a population living in a constant environment: "As is clear by now, equilibrium models presume a constant environment" (Reice 2001:17) (do not confound demographic close population and biological constant environment with genetic assumption of close population). The population comparisons as intended by Lotka and Fisher, using the Net RV of the population in its steady state, can include migration rates, although calculations become harder. Furthermore, the tool of RV can be extended to take into account migration rates. Therefore, the demographic assumption of ceteris paribus population is a constant environment population assumption, and the steady states to which this population tends towards can be stationary or stable states.

Not having a well-defined closed population assumption has led to confusions in demography, and some of them have not been clarified. To avoid these confusions in subsequent uses of RV and stable population theory is important to emphasize this point: it is obvious that the ceteris paribus population assumption is an astringent simplification of reality; populations are open systems because the environment (everything else) is changing and so demographic rates will not remain constant for the period of time needed to achieve a steady state. Populations are more likely to be found in transient states (which are moving towards steady states but the steady state is almost never reached). Even Fisher realized that the steady state is only an hypothetical one "Actual populations are seldom at or near the steady state appropriate to their birth and death rates, and could scarcely become so unless the frequency at each age of death and reproduction remained constant for nearly a century" (Fisher 1927:106).

Therefore the stable equivalent population is only a hypothetically equivalent population and its age distribution and growth rate are not 'true', 'future' or 'ultimate' characteristics, they are only hypothetic ones. This may seem quite obvious but several influential papers have confused demographers over quite long time, and it is important to revisit these confusions in order to prevent further misinterpretations. For example, Lotka (1925) provides the first example of misinterpretation when he named the stable growth rate as the 'true' population growth rate; Goodman (1967) also made the same mistake when he named it as 'ultimate' or 'eventual' population growth rate. The stable growth rate is nothing more than the hypothetical rate that follows from the astringent assumption of constant rates. This stable growth rate is a useful decomposition of the actual growth rate (it is 'free' from the effects of the actual age distribution) but if there is any *true* rate of population growth it is the actual or transient growth rate. That is why some researchers are now being more critical with assuming stable characteristics for the populations. For example Koons et al (2005) explain that asymptotic demographic analysis has had a long history of use in population studies, however the stable population state should not be assumed unless empirically justified; and they suggest taking special consideration of the actual population growth rate, which they call *transient growth rate* as opposed to the *stable* or intrinsic growth rate. One corollary of this discussion is that the population comparisons using the Net RV of the stable population, as intended by Lotka and Fisher, are not comparisons of the 'true' or 'eventual' behaviour of the population, they are only comparisons of hypothetic populations that would emerge from constant formation and decomposition rates. Keyfitz and Flieger also provide a good example of misinterpretation of RV and stability: "*V* Reproductive value: the expected future girl children who will be born to the existing female population on the observed regime of mortality and fertility, discounted at the intrinsic rate of natural increase" (Keyfitz and Flieger 1968:22). The net reproductive value *V* is not the *expected future* children who *will* be born; it is the *hypothetic* children who *would* be born if the observed vital rates were held constant for very long periods of time (and is obvious that the rates will not remain constant for long periods of time). So the comparisons of populations in their steady states using RV, as intended by Lotka and Fisher and done by Keyfitz and Flieger, are not comparisons of 'future' or 'eventual' states of the populations, they are just comparisons of *hypothetic* populations that would emerge from the *unrealistic* assumption of a *ceteris paribus* population. Another corollary of this discussion is that assumptions are of great importance in population analysis, especially when it comes to interpreting results. Taking these clarifications into account is how we should understand the population comparisons suggested by Lotka and Fisher.

OTHER STABLE VALUATIONS

It is also important to revisit another source of confusion. What Fisher (1927) wrote is that, if we want to compare the reproductive value of different populations (or the same population in different time points) it is not suitable to use the arbitrary assumption of v_0=1. Instead is better to use another arbitrary assumption for v_0, namely the one that allows V, the Net RV, of the stable population to be equal to the size of this hypothetical population (such arbitrary assumption is given by equation 9). When Lotka read Fisher's (1927) paper he realized that he had already proposed this idea along with his stable population theory, later (Lotka 1928, 1939, 1942) he formalized the fertility relationships in stable populations. Within these relationships Lotka introduced the *constants* Q, which are the roots of the so called 'renewal equation'. Only one of these roots is real and is associated with the stable growth rate. In his papers Lotka used several different notations for this root, in his book (1939) he named it Q_ρ and he used it to calculate a constant K, which he used to calculate the size of the stable population. Later Keyfitz (1939) named this constant K as 'stable equivalent value' but he denoted it by Q. Keyfitz also proved that "In fact, V [with conventional valuation v_0=1] is a simple multiple of Q. In the continuous representation, V is exactly equal to Q multiplied by the intrinsic birth rate b and by the mean age at childbearing" (Keyfitz and Caswell 2005:204). In a private communication Crow explained, "The main difference, I think, is in the implications, not the algebra. Lotka and Keyfitz were thinking of demographic problems; Fisher was thinking more broadly of evolutionary problems."

Revisiting the changes in the notation is important because is easy to get confused between the *real root* of the 'renewal equation' and the *stable equivalent value*. The stable equivalent value Q equals V with the arbitrary assumption of equation 9, but the real root of the 'renewal equation' denoted by Keyfitz as Q_1 is V with the following arbitrary assumption:

$$v_0 = \frac{1}{x_r^s} \quad , \tag{10}$$

this new valuation is the one used by Goodman "the reciprocal of v_0 is the average age of the mothers of all those who are in the 0^{th} age-interval in the stable population, i.e. the 'average age at childbirth' in the stable population" (Goodman 1967:544); and is useful to calculate the hypothetical contribution of the different age groups to the stable equivalent population.

So it is important to realize that Q and Q_1 are different concepts: Q is referred as the stable equivalent value; and the real root Q_1 as the stable equivalent births. Both Q and Q_1 are V with different conventional valuations for v_0. If Keyfitz was unaware of the original paper of Fisher (1927) and the letter from Lotka (1927), then he rediscovered the content of Lotka's letter: the main ideas of Fisher 1927 had already been posed in Lotka's stable population theory. Furthermore, if Fisher never explained to Keyfitz his original ideas for RV (even though Fisher was Keyfitz's professor at North Carolina State University in 1946, and there is evidence in Bennett 1990 that they exchanged correspondence), then Keyfitz rediscovered two of the main points of Fisher (1927): the main utility of RV rests in the non-absolute valuation of v_0, and the conventional valuation given by equation 9 is useful for comparing populations in their *steady state*. Even terminology is confusing, in his early papers Keyfitz named Q as the stable equivalent number or stable equivalent population (see Keyfitz 1971a; this term confused me when reading Crow 2002 by the first time), but this name can be easily mixed up with the broader concept of stable 'standard' population (as Lotka named it) or ultimate stationary population (as Keyfitz first named it). It seems that Keyfitz realized this problem, because in his later books he referred to Q as stable equivalent value (see Keyfitz 1985) and used the name of stable equivalent population for Lotka's stable 'standard' population (the specific stable population which can be uniquely associated to a given set of vital rates and age distribution).

This brief history of the quantities V and Q may seem just as a mathematical curiosity, but it has had major influence in demography. Keyfitz (1971a) used V with the arbitrary valuations given by equations 10 and 9, which he called stable equivalent births Q_1 and stable equivalent of the total population Q_0 (in his latter books he denoted it only by Q), to develop the concept of *population momentum* for an immediate drop of fertility to replacement level

$$\frac{Q}{N_0} = \frac{b^s \overline{x_d^s}}{r \overline{\overline{x_r^s}}} \left(\frac{R_0 - 1}{R_0} \right) \quad , \tag{11}$$

where Q is the stable equivalent value, N_0 is the size of the initial population, r and b^s are the stable growth and birth rates, x denotes age so in the upper part there is the stable mean age at death (life expectancy) and in the lower part the stable mean age at reproduction (childbearing), finally R_0 denotes the net reproductive rate. The population momentum is clearly a multiple of the stable equivalent value, so it is also V with a different conventional valuation, namely

$$v_0 = \frac{1}{N_0 b^s \overline{\overline{x_r^s}}} \quad . \tag{12}$$

Besides the importance of this concept for mathematical demography, the influential paper about *population momentum* was clearly written to support already existing birth control and family planning programs: "In some countries hesitation in making contraception available is rationalized by the view that the country is not yet 'full'. Governments as far separated as those of Brazil today and of Indonesia in Sukarno's time refer to plentiful land and other resources as evidence that their populations could stand considerable further increase before they must become stationary. Concern that total numbers will taper off prematurely is misplaced. If presently high-fertility countries were to experience an immediate drop to age-specific birth rates that would just replace existing parents, the ultimate stationary population would be about two thirds higher than the present total" (Keyfitz 1971a:71). So the rediscovery of the non-absolute valuation for newborns RV has been of importance also in applied demography and in supporting population policies (for the influence of these programs and policies on demography see Caldwell and Caldwell 1986).

However, confusion still remains. For example, the paper of Keyfitz (1971a) presents misleading terms and assumptions like ‘ultimate size’, ‘must become stationary’, and so on (problems obviously derived from misinterpretations of closed and open assumptions leading to steady states of the population). In this sense it is important to notice that the idea (and the formula) of *population momentum* was already stated in Fisher 1927. And Fisher’s interpretation of formula 11 is more appropriate because he was not thinking about ‘eventual’ or ‘ultimate’ states of the population but about comparisons with potential or hypothetic states: “the comparison of the census population with its total value [V calculated using equation 9] provides a simple index showing to what extent the actual populations are or are not of ages favourable to reproduction” (Fisher 1927:106-107).

FURTHER USES OF RV

Besides the subtleties of actual subpopulation comparisons, RV has other interesting uses: “an epidemic killing largely persons beyond the reproductive age may have little direct influence upon the population of future generations; nevertheless, in the crude statistics which depend upon total deaths it may loom larger than a war which destroys a large proportion of the age groups of early manhood… The total value [V^s] on this system of the population of this island [England] is probably somewhat above its census total, and it is quite possible that while the total value for reproductive purposes is decreasing, the number of heads is increasing by a kind of transference from potential to actual humanity” (Fisher 1927:105-107). Fisher, being one of the main supporters of the eugenics movement (Fisher 1914), was fixated with declining populations and ‘race’ suicide, so he insisted in using RV as a tool for analyzing reproduction ‘potentials’ of actual populations. Despite Fisher’s obsessions, his phrase remarks the importance of transient states and population momentum: rates are not constant over time and comparing actual versus hypothetic stable populations is of interest in demographic analysis.

The main idea is to use RV as a tool for analyzing how environmental disturbances affect the relation among actual and stable (potential) populations. There are good examples of these analyses in Biology, for example Michod (1979) investigated how withdrawals of RV from specific age-classes with increased mortality would tend to stabilize total population size and guard against extinction; Koons et al. (2005) stressed the importance of the transient growth rate given environmental disturbances, and they concluded that variation in a population’s initial Net RV largely explained the variation in transient growth rates and their sensitivities to changes in life-cycle parameters. They explain that the initial Net RV is an ‘omnibus’ measure that can be used to predict the transient dynamics across the initial state conditions, through time, and to examine shifts in the rank-order of vital rate contributions to transient growth rate. In Demography perhaps the best examples of studying disturbances and RV are given in Keyfitz’s books: “A single theory [RV] answers questions about the numerical effect of sterilization, mortality, and emigration, all supposed to be taking place at a particular age *x*. By means of the theory we will be able to compare the demographic results of eradicating a disease that affects the death rate at young ages, say malaria, as against another that affects the death rate at older ages, say heart disease” (Keyfitz 1985:142).

Recently, while analyzing important environmental disturbances on human populations (massive deportations to Siberia in Russian population and massive immigration in US population), Ediev (1999, 2000, 2001a, 2001b) proposed the concept of *demographic potential*. With this concept he rediscovered the importance of non-absolute valuations for RV. Ediev

(2001a) explains that he developed this concept while trying to estimate population's demographic losses caused by external factors; these losses couldn't be directly estimated due to the lack of data so he had to solve this problem without using population projections at all. He thought about an index that changes as a function of time when a population reproduces itself without 'external' influence, but he recognizes that distinguishing between 'external' and 'intrinsic' factors is a matter of model assumptions, i.e. is a matter of taste. Ediev proposed that, if only one sex is concerned, the age specific demographic potentials denoted by c are given by

$$c(x,t)=\int_x^{\infty}\frac{l(u,t)}{l(x,t)}b(u,t)c(0,t+u)du \quad . \tag{13}$$

Ediev also shows that, when fertility and mortality are assumed to be constant over time, the demographic potentials decrease as an exponential function of time and are equal to Fisher's reproductive value if the demographic potential of a newborn is set to be one and other potentials are expressed in terms of the newborn's potential

$$\frac{c(x,t-x)}{c(0,t)}=c(x)=\frac{c(0)}{e^{-rx}l(x)}\int_x^{\infty}l(u)b(u)e^{-ru}du \quad . \tag{14}$$

Ediev regards his *demographic potential* as a generalization of RV, but as we have seen, equation 14 is exactly the same as Fisher's 1927 original presentation of RV (see equation 6). Furthermore, as has been emphasized and it should be obvious now, the convention of setting the 'demographic potential' of a newborn equal to one is only an arbitrary assumption; and Fisher (1927) gave no absolute value for a newborn because he knew that other conventional valuations are also useful. So Ediev's demographic potential of population, when applied to one sex, is not a generalization of RV, it is *exactly* RV. But when Ediev (2000) computes demographic potentials with both sexes included he is indeed generalizing the concept of RV. Besides using this concept for calculating ‘external’ population losses in Russia and doing backward and forward projections, Ediev (2001b) also reconstructs US immigration history, so he proposes several interesting analyses that can be done with RV or, as he names it, *demographic potential*.

This is not a biological paper but is still important to note that RV has many other uses in biology and ecology. Fisher’s RV gives: a way of assessing the effect of a specific age group on a hypothetically future population and therefore the importance of this age group for evolutionary analysis; a weighting system to smooth out uncertainties in the actual age distribution for various genetic applications, in particular Fisher’s conjecture known as ‘Fundamental Theorem of Natural’ selection (see Crow 1979).

MATRIX CONTEXT AND INTERPRETATION

Perhaps the easiest way to understand RV and the roll of arbitrary valuations is using Leslie matrices. As Keyfitz and Caswell (2005) explain, the right eigenvector $\mathbf{w}_1$ of the transition matrix is associated to the stable population growth rate given by the dominant eigenvalue, and this vector yields the stable age distribution of the population when scaled to sum 1; the left eigenvector $\mathbf{v}_1$ gives the age specific reproductive values, “if we take ‘the contribution of stage *i* to long-term population size’ as a reasonable measure of the ‘value of stage *i*’, the left eigenvector $\mathbf{v}_1$ gives the relative reproductive values of the stages. We must insert the qualifier ‘relative’ because eigenvectors can be scaled by any nonzero constant [and they still are eigenvectors]… It is customary to scale $\mathbf{v}_1$ so that its first entry is 1” (Keyfitz and Caswell 2005:209). The use of eigenvectors clarifies the idea of conventional valuations; each different valuation corresponds to a different scaling but the eigenvector is not altered by the scalar multiplication. When comparing the RV of different ages (or stages) within a population the eigenvector should be scaled to have the valuation v_0=1. When comparing different populations (or the same population on different time periods), as Fisher (1927) suggested, v_0 must be chosen so that the total reproductive value is equal to the population size in the steady state; in matrix terminology, eigenvector $\mathbf{v}_1$ should be scaled so V equals the size of the stable equivalent population. This is the same scaling needed to obtain Q "That is, the stable equivalent is just the total reproductive value of the initial population, when scaled so that $\|\mathbf{w}_1\| = 1$ and $\mathbf{v}_1 \bullet \mathbf{w}_1 = 1$" (Keyfitz and Caswell 2005:213).

Let N^s be the size of the stable equivalent population and V^s be its total reproductive value, then V^s is the scalar product of $\mathbf{v}_1$ and $\mathbf{w}_1$ (note that $\mathbf{v}_1$ is a row vector and $\mathbf{w}_1$ is a column vector). To obtain V^s=N^s we need the following scaling

$$\|\mathbf{w}_1\| = \mathbf{1w}_1 = 1 \qquad \text{and} \qquad \mathbf{v}_1 \bullet \mathbf{w}_1 = 1 \;,$$

$$\Rightarrow \qquad V^s = \mathbf{v}_1 \bullet (N^s \mathbf{w}_1) = N^s(\mathbf{v}_1 \bullet \mathbf{w}_1) = N^s \qquad .\ QED.$$

RV is the left eigenvector $\mathbf{v}_1$ and it can be scaled according different conventional valuations of v_0. Therefore, the quantities Q, Q_1 and 'population momentum' are simply the results of different scaling of $\mathbf{v}_1$, they are particular cases of the general concept of net reproductive value V. Because of the eigenvector properties, the concept of RV does not change with each arbitrary assumption or scaling; but the analysis and the interpretations of RV are highly dependent on the assumptions used to scale the eigenvector. This dependence on arbitrary assumptions is clearly important and cannot be stressed enough.

In the matrix context is evident that the scaling of RV is a matter of convenience, and when comparing subpopulations is 'convenient' to consider generation time. "When the matrices A and A^* are different, then the effects of the difference in the average age at childbirth under the two regimes, A and A^*, must be taken into account" (Goodman 1967:545). Because it is difficult to know when a specific valuation is more 'convenient', is always important to remember that:

i) RV is an *average* measure. The unit of study when using RV is the population (or arbitrary subpopulations) not the individuals. It is a mistake to ascribe a RV to single individuals like a woman or a migrant (we must remember that any individual can have several reasons for not engaging in reproduction: infertility, personal decision, and etcetera). In a high level of abstraction we could associate RV to an 'average' individual, but we must remember that what is called 'average' individual, the population statistical mean, might not even exist in the actual population. So it seems better to avoid at all using misleading terms as 'average' individual.

ii) When comparing 'different' populations we must remember that all divisions of human populations are arbitrary, and the assumptions made when defining 'different' populations should be stated. The age-classification or age-division of a population is nothing more than an *arbitrary* division (for example, we can use single ages, 5-year age groups; instead of using discrete ages we could work with continuous ones). Just as the age-division of a population, all other divisions of the human population are taken from *arbitrary assumptions* that should be stated: a very dull but obvious case is Fisher's 1927 division of human population in 'working classes' given 'genetic' differences (we must remember that even genetic research does not lead to 'natural' subdivisions of human population, Cavalli-Sforza et al. 1997; Owens and King 1999). When comparing 'potential reproduction' of migrants and non-migrants, citizens and non-citizens, or any other arbitrary division of the population, we must remember and state clearly that we are imposing 'differences' among the individuals given by arbitrary laws or rules like country of origin, language, etc. In other words, any comparison of 'different' populations can only be made if we assume that there are differences among individuals, and this assumption has little to do with scientific thinking but a lot to do with politics and human arbitrary constructions.

DISCUSSION

Fisher never intended v_0 to have a fixed value, he was well aware that the utility of his tool depended on its flexibility. Different conventions on the value of v_0 can be used to answer diverse research questions. Because there still no generally agreed notation, confusion may arise; therefore when using RV the valuation of v_0 must always be specified. Table 1 shows several conventions and their related Net RV.

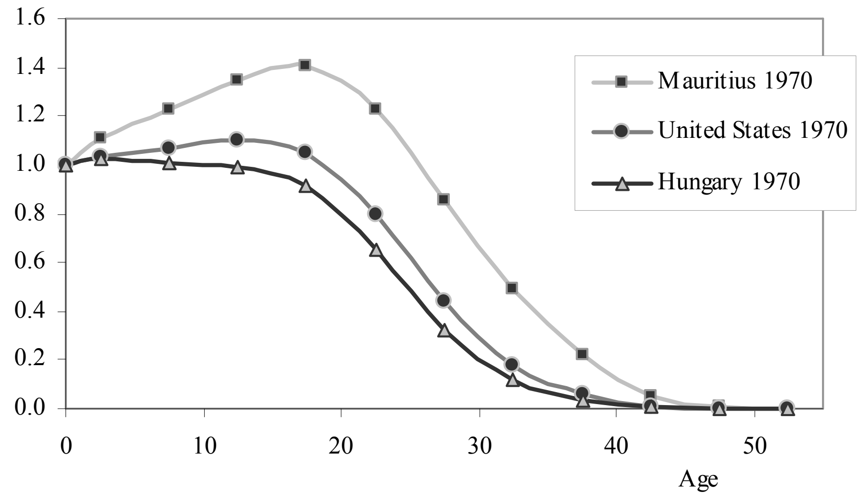
Table 1. Conventional valuations of v_0 and related Net Reproductive Value.

v_0	$V = \int_0^\infty v_x n_x dx$	**Intended use**
$v_0 = 1$	'usual' V	Age or classes comparisons
$v_0 = 1/\overline{x_r^s}$	$V = Q_1$ Stable equivalent births	Stable reproduction comparisons
$v_0 = 1/\overline{b^s}\,\overline{x_r^s}$	$V = Q$ Stable equivalent value	Subpopulations comparisons
$v_0 = 1/N_0\overline{b^s}\,\overline{x_r^s}$	V = Population momentum	Actual versus stable comparisons

Fisher (1927) and several other researchers (Lotka 1925; Hamilton 1966; Leslie 1966; Goodman 1967) noted that the effect of dissimilar generation times must be taken into account when comparing reproductive values of different subpopulations. Even though the paper of Crow (2002) has already called for a revision of the 'usual' valuation of RV, there is still confusion about the use of arbitrary assumptions in RV (some are clearly in contradiction with the original paper of Fisher 1927). For example, when explaining reproductive value Keyfitz and Caswell (2005) compare Mauritius, United States and Hungary for years close to 1970 (they use data from Keyfitz and Flieger 1971); but they apply the arbitrary assumption v_0=1, which Fisher described as inappropriate for population comparisons. The comparison made in Keyfitz and Caswell 2005, for the same countries, for the year 1970 is shown in Figure 1 (data from Keyfitz and Flieger

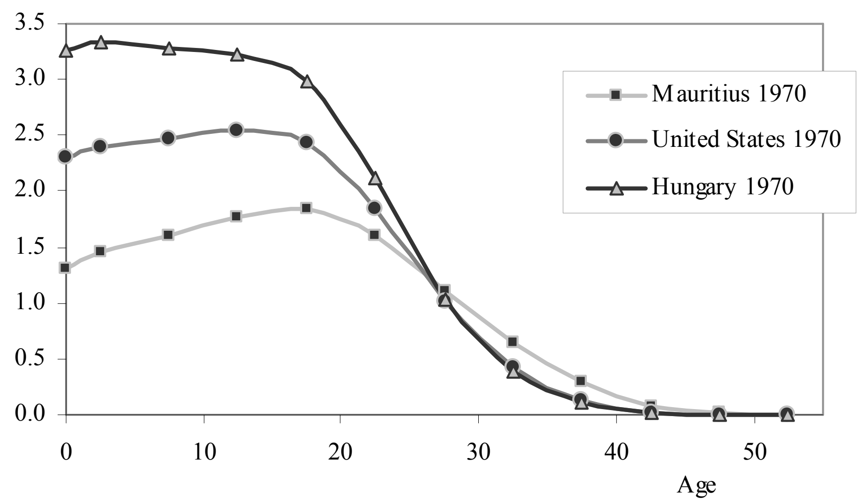
1990). On the other hand, Figure 2 shows the same comparison with proper assumption of equation 9. Fisher's reasons for this suggestion can be summed up in the following quote: "The convention of valuation based upon the steady state allows the new born child to count for more among long lived people [aging population] than among the short lived [young population], as he obviously ought to do" (Fisher 1927:106).

Figure 1, Misleading comparison of Reproductive Values (as shown in Keyfitz and Caswell 2005).



Source: Own elaboration, data from Keyfitz and Flieger (1990).

Figure 2, Appropriate comparison of RV taking generation time into account.



Source: Own elaboration, data from Keyfitz and Flieger (1990)

Reproductive value is a useful tool. As Keyfitz (1985) explains, it can be used to answer question about the numerical effects of sterilization, mortality and emigration at particular ages; it also allows demographic analysis of eradicating specific diseases. Furthermore, researchers like Michod (1979), Koons et al. (2005) and Ediev (2001) are proposing new uses of RV for understanding evolutionary mechanisms and population history and prospects. But when looking Figures 1 and 2, it becomes obvious that the effect of the comparisons and interpretations of RV are highly dependent on the conventional valuations of v_0. So we should not forget that conclusions taken from RV are highly dependent on the assumptions made during RV calculations. Keeping this in mind becomes even more important when we remember that Fisher (1914) was a leading eugenist and that his quantitative conclusions about the reproductive potential of societies (and about humankind in general) were faulty because of his erroneous assumptions on arbitrary population divisions. Therefore, the most important point to clarify about RV is that when calculated on an objective scale, the measure of reproductive value implies absolutely nothing about social values, 'racial' superiorities, etc.

ANNEX: Lotka's 1927 letter.

CORRESPONDENCE.

THE EDITOR OF EUGENICS REVIEW,

Dear Sir,—

It is with much interest that I have read Dr. R. A. Fisher's article "The Actuarial Treatment of Official Birth Records" in the July issue of EUGENICS. May I be permitted to draw attention through your columns to certain anticipations both of the general principle and of certain points of detail, of Dr. Fisher's application of actuarial relations to birthrates.

The formula by which he proposes to compute the "Malthusian" rate of geometrical increase was given by Sharpe and Lotka in the *Philosophical Magazine* for April, 1911, page 485; I have also referred to it repeatedly in later publications. Quite lately it has been independently given by J. B. S. Haldane in the *Proceedings of the Cambridge Philosophical Society*.

Dr. Fisher remarks: "The exact calculation of rate of population increase or decrease freed from the *temporary* and *adventitious* elements of the *momentary* age distribution . . . . opens the way to the study of a whole new aspect of human vital statistics." In my book *Elements of Physical Biology*, 1925, page 115, after discussing precisely these principles, I have remarked: "This element (i.e., the influence of irregularities in the age distribution) is . . . automatically ruled out of further discussion; a circumstance which is in so far fortunate as we are here interested in the relation of the rate of growth to the more fundamental biological characteristics; the age distribution appearing merely as an *adventitious* element complicating the relation, without being essential to the fundamental characterisation of the species." The principle is also referred to in a footnote in an article by Dublin and Lotka, Journal of the American Statistical Association, September, 1925, page 306, as follows: "Perhaps it may be well to draw attention to the fact that the difference between the existing state of affairs and that here serving as a basis of comparison, lies merely in the *adventitious* factor of age distribution. The essential, fundamentally biological factors are strictly the same in the actual and in the standard of comparison. And whereas the existing age distribution represents a *transitory* state of affairs incapable of continued existence under constant conditions of maternity frequency and mortality, the age distribution of the standard of comparison is that which is stable, self-sustaining, under the biological conditions *prevailing* now.

Again, Dr. Fisher remarks: "For example, an epidemic killing largely persons beyond the reproduction age may have little influence upon the population of future generations. . . . Simply and solely from the point of view of the balancing of death by reproduction these lives are not of equal value. . . ." To this deduction, which follows directly from the method under discussion, I have drawn attention in the following words (Physical Biology, p. 119): *Effect of Selective Slaughtering.* "A second deduction from equation (34) follows from the fact that the integral extends only between the age limits of the reproductive period. In consequence of this the rate of increase, per head, of the population is independent of the form of the life curve (the survival factor) outside this period; thus the slaughtering of the "superannuated" members of a herd has no effect upon its rate of increase, though both death rate and birth rate are augmented (by equal amounts). The economic significance of this, as a means for raising the productive capacity of the herd, considered as a food factory, is self-evident."

The same principle, applied to a human population, was fully set forth by me in a paper, *The Progressive Adjustment of Age Distribution to Fecundity*, in the Journal of the Washington Academy of Science, November, 1926, p. 505. Computations and curves built on actual numerical data for the United States were exhibited, showing separately the rate of increase computed on the basis of the female population under fifty-five years of age alone, thus demonstrating the proposition that this rate is fully determined without any reference to the female population at higher ages. These computations are developments of a paragraph which first appeared in my article *A Natural Population Norm*, Journal of the Washington Academy of Science, 1913, p. 293.

That the actuarial principle discussed by Dr. Fisher may be used as an instrument to measure effective fertility is fully set forth, with numerical application, in my paper: *The Measure of Net Fertility*, *Journal of the Washington Academy of Science*. December, 1925, page 469.

A particularly detailed application, with much numerical elaboration, of the principle under discussion, has been given by Dublin and Lotka in the paper referred to above, *The True Rate of Natural Increase*. Among other things brought out and illustrated by actual numerical computations in this paper is the separate application of the principle to the female and the male population, a point to which Dr. Fisher also refers. A further application is to be found in my article, *The Size of American Families in the Eighteenth Century*, Journal of the American Statistical Association, June, 1927, page 153. Other work is in progress.

It will thus be seen that the subject of the application of actuarial principles to birthrates has progressed well beyond the point of a project with "future" possibilities. It is an actuality.

Very truly yours,
ALFRED J. LOTKA.

ED. NOTE.—Dr. Fisher writes: I am much interested to see how closely Dr. Lotka's work, which I had not previously seen, agrees in aim and method with the recommendations I have made. Evidently the only absolutely novel suggestion in my article lies in the estimation of a definite "reproduction value" for each age of life. Dr. Lotka's suggestions and mine are still unfortunately in the future as far as British official birth data are concerned.

REFERENCES

Bennett, H. 1990. *Statistical inference and analysis: Selected correspondence of R.A. Fisher*. Edited by J. H. Bennett. Claredon Press, Oxford.

Caldwell, J. and P. Caldwell. 1986. *Limiting Population Growth, and the Ford Foundation Contribution*. Frances Pinter, London.

Caswell, H. 2001. *Matrix Population Models*. Sinauer Associates, Inc., Massachusetts.

Cavalli-Sforza, L. et al. 1997. "An apportionment of human DNA diversity". *Proceedings of the National Academy of Sciences*, 94:4516-4519.

Cerone, P. 1970. "On Stable Population Theory with Immigration". *Demography*, 24(3):431-438.

Crow, J. 1979. "Gene frequency and fitness change in an age-structured population". *Annals of Human Genetics*, 42(3):355-70.

Crow, J. 2000. "Fisher's contributions to genetics and evolution". *Theoretical Population Biology*, 42(3):355-70.

Crow, J. 2002. "Perspective: Here's to Fisher, Additive Genetic Variance, and the Fundamental Theorem of Natural Selection". *Evolution: International Journal of Organic Evolution*, 56(7):1313-1316.

Ediev, D. 1999. *Demographic and economic-demographic potentials* (in Russian). Ph. D. thesis, Moscow Institute of Physics and Technology, Moscow.

Ediev, D. 2000. "Principles of Aggregate Demographic-Economic Modeling Based on Demographic Potential's Technique". Presented in the *Workshop on Demographic-Macroeconomic Modeling*, Max Planck Institute for Demographic Research, Rostock.

Ediev, D. 2001a. Application of the Demographic Potential Concept to Understanding the Russian Population History and Prospects: 1897-2100. *Demographic Research*, 4:289-335.

Ediev, D. 2001b. “Reconstruction of the US immigration history: demographic potential approach”. *Investigated in Russia*, 140:1619-1635.

Feeney, G. 1970. “Stable Age by Region Distributions”. *Demography*, 7(3):341-348.

Fisher, R. 1914. “Some Hopes of a Eugenist”. *Eugenics Review*, 5:309-315.

Fisher, R. 1927. “The Actuarial Treatment of Official Birth Records”. *Eugenics Review*, 19:103-108.

Fisher, R. 1930. *The Genetical Theory of Natural Selection*. Reprinted by Dover in 1958, New York.

Fisher, R. 1965. *Experiments in plant hybridisation*. Oliver & Boyd, Edinburgh.

Fisher, R. 1999. *The genetical theory of natural selection: a complete variorum edition*. Edited with an introduction and notes by H. Bennett. Oxford Univ. Press, Oxford.

Goodman, L. 1967. “On the Reconciliation of Mathematical Theories of Population Growth”. *Journal of the Royal Statistical Society*, 130(4): 541-553.

Haldane, J. 1927. “A mathematical theory of natural and artificial selection, V, selection and mutation”, *Proc. Camb. Philos. Soc.*, 23:838-844.

Hamilton, W. 1966. “The Moulding of Senescence by Natural Selection”. *Journal of Theoretical Biology*, 12:12-45.

Haynie, D. 2001. *Biological Thermodynamics*. Cambridge University Press, Cambridge

Humphreys, N. ed. 1885. *Vital statistics, a memorial volume of selections from the reports and writings of William Farr*. London: Sanitary Institute. Reprinted in 1975 by Scarecrow Press.

Keizer, J. 1984. “Reversibility, work, and heat at nonequilibrium steady states”. *Physical Review*, 30(2):1115-1117.

Keyfitz, N. 1968. *Introduction to mathematics of population*. Addison-Wesley, Reading, Massachusetts.

Keyfitz, N. 1969. “Age distribution and stable equivalent”. *Demography*, 6:261-269.

Keyfitz, N. 1971a. “On the Momentum of Population Growth”. *Demography*, 8:71-80.

Keyfitz, N. 1971b. “Migration as a Means of Population Control”. *Population Studies*, 25(1):63-72.

Keyfitz, N. 1985. *Applied Mathematical Demography*. Springer, New York.

Keyfitz, N. and H. Caswell. 2005. *Applied Mathematical Demography*. Springer, New York.

Keyfitz, N. and W. Flieger.1968. *World Population: An Analysis of Vital Data*. University of Chicago Press, Chicago.

Keyfitz, N. and W. Flieger. 1971. *Population: Facts and Methods of Demography*. W. H. Freeman, San Francisco.

Keyfitz, N. and W. Flieger. 1990. *World Population Growth and Aging: demographic trends in the late twentieth century*. The University of Chicago Press, Chicago.

Koons, D. et al. (2005). “Transient population dynamics: relations to life history and initial population state”. *Ecological Modelling*, 185:283-297.

Leslie, P (1945). “On the Use of Matrices in certain Population Mathematics”. *Biometrika*, 33:183-212.

Leslie, P.H. (1948), “Some Further Notes on the Use of Matrices in Population Mathematics”, *Biometrika*, 35:213-245.

Leslie, P. 1966. “The intrinsic rate of increase and the overlap of successive generations in a population of guillemots (Uria aalge Pont.)”. *Journal of Animal Ecology*, 35:291-301.

Lotka, A. 1921. “Note on Moving Equilibria”. *Proceedings of the National Academy of Sciences*, 7:168-172.

Lotka, A. 1927. “Correspondence between Alfred James Lotka and R. A. Fisher regarding the article ‘the actuarial treatment of official birth records’ by R. A. Fisher”. *Eugenics Review* , 19 (1927):257-258.

Lotka, A. 1925. “On the True Rate of Natural Increase”. *Journal of the American Statistical Association*, 20(151):305-339.

Lotka, A. 1928. “The progeny of a population element”. *The American journal of hygiene*, 8:875-901.

Lotka, A. 1939. *Théorie analytique des associations biologiques*. Volumes I and II. Hermann et cie., Paris.

Lotka, A. 1942. “The progeny of an entire population”. *The annals of mathematical statistics*, 13:115-126

Lotka, A. 1944. “Comparison of two methods of estimating capitalized value of earning capacity”. *Journal of the Washington Academy of Sciences*, 34(1):10-13.

Lotka, A. 1969. *Teoría analítica de las asociaciones biológicas*. Centro Latinoamericano de Demografía CELADE, Chile.

Lotka, A. 1998. *Analytical Theory of Biological Populations*. Springer, USA.

Medawar, P. 1957. *The Uniqueness of the Individual*. Methuen & Co., Great Britain.

Michod, R. 1979. “Evolution of Life Histories in response to Age-specific Mortality Factors”. *The American Naturalist*, 113(4):531-550.

Mirowski, P. and C. Goodwin. 1991. *More Heat than Light : Economics as Social Physics, Physics as Nature's Economics*. Cambridge University Press, Cambridge.

Owens, K. and MC. King. 1999. “Genomic Views of Human History”. *Science*, 286(5439):451-453.

Reice, S. 2001. T*he Silver Lining: The Benefits of Natural Disasters*. Princeton University Press, Princeton.

Samuelson, P. 1977, 1978. “Generalizing Fisher’s ‘reproductive value’”, *Proceedings National Academy of Sciences*, USA. I. 74: 5189-5192; II. 74: 5772-5775; III. 75: 4062-4066; IV, 75: 6327-6331.

Sharpe, F. and A. Lotka. 1911. “A problem in the age-distribution”. *Philosophers Magazine*, 6(21):435-438.

Smith, E. and D. Foley. 2002. “Is utility theory so different from thermodynamics?”. Working paper, Santa Fe Institute, Santa Fe.

Stearns, S. 1976. “Life-History Tactics: a review of ideas”. *The Quarterly Review of Biology*, 51(1):3-47.

Stigler, S. 2005. “Fisher in 1921”. *Statistical Scientist*, 20(1):32-49.

Zotin, A. 1990. *Thermodynamic Bases of Biological Processes*. Walter de Gruyter, New York.